Surface Integrals (cont.)

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Riemann sum: break S into pieces $S_1, \dots, S_N$

$$\sum_{i=1}^{N} f(x_i, y_i, z_i) \cdot \text{area}(S_i) \quad \text{s.t.} \quad (x_i, y_i, z_i) \in S_i$$

<u>via Parameterization</u> $(x(s,t),\ y(s,t),\ z(s,t))$ for $(s,t) \in [a,b] \times [c,d]$

$$\iint f\, dA = \int_c^d \int_a^b f(x(s,t), y, z) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| ds\, dt$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\vec{k} = (x, y, z)$$

Integral of a vector fcn over a surface

$$\iint_S \vec{F}\, d(\text{something}) = \int_c^d \int_a^b \det \begin{vmatrix} \vec{F} \\ \partial \vec{r}/\partial s \\ \partial \vec{r}/\partial t \end{vmatrix} ds\, dt$$

$$\det \begin{vmatrix} \vec{F} \\ \partial \vec{r}/\partial s \\ \partial \vec{r}/\partial t \end{vmatrix} = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right)$$

$$= \left(\vec{F} \cdot \left[\frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|} \right] \right) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|$$

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|}$$

← unit vector $\perp$ to S

<u>unit normal vector</u>

$\Rightarrow \iint_S (\vec{F} \cdot \vec{n})\, dA$ is how we take an integral of a vector fcn

<u>Notice</u> if $\vec{F}$ always $\perp$ to S, then $\vec{F} \cdot \vec{n} = \|\vec{F}\|$

↳ in this case, surface integral of $\vec{F}$ is just $\iint \|\vec{F}\|\, dA$

What is orientation?

For line integrals, parameterization determines orientation

Idea: direction is from small t to larger t.

$(x(t), y(t)) \quad a \leq t \leq b$

reverse orientation $\quad (x(b+a-t), y(b+a-t))$

reverses orientation

In computing line integral, $x'(t), y'(t)$ get negated.

For surface integrals, we have the factor

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$$

$$= -\left(\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial s}\right)$$

↳ To switch orientation of S, switch s & t

→ this should negate normal vector.

→ why?

RHR (right hand rule)

algebraic explanation

matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ switches the 2 coordinates and has det -1

$\Rightarrow$ this matrix reverses orientation

e.g. of orientations on surfaces

sphere

↳ inward & outward pointing

Given a parameterization, how to know the way it's pointing?

$$S = \{x^2 + y^2 + z^2 = 1\}$$

~~For~~ $0 \leq s, t \leq \frac{1}{2}$:

$$(x, y, z) = (s, t, -\sqrt{1 - s^2 - t^2})$$

intuitively, (s, t) in same direction as (x, t)

$\Rightarrow$ RHR say $\vec{n}$ points up $\Rightarrow$ inward

(x, t)

$\Rightarrow$ RHR say $\vec{n}$ points up $\Rightarrow$ inward

algebraically

$$\frac{\partial \vec{r}}{\partial s} = \left(1,\ 0,\ -\tfrac{1}{2}(1-s^2-t^2)^{-\frac{1}{2}}(-2s)\right)$$

$$= \left(1,\ 0,\ \frac{s}{\sqrt{1-s^2-t^2}}\right)$$

$$\frac{\partial \vec{r}}{\partial t} = \left(0,\ 1,\ \frac{t}{\sqrt{1-s^2-t^2}}\right)$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \left(\frac{-s}{\sqrt{1-s^2-t^2}},\ \frac{-t}{\sqrt{1-s^2-t^2}},\ 1\right)$$

this points up from bottom of sphere $\Rightarrow$ inward

Plane (eg yz plane)

$\hookrightarrow$ 2 possible orientations are $+x$ direction ? $-x$ direction

ex: $(x,y,z) = (0,s,t)$

$$\frac{\partial \vec{r}}{\partial s} = (0,1,0) \qquad \frac{\partial \vec{r}}{\partial t} = (0,0,1)$$

and $(0,1,0) \times (0,0,1) = (1,0,0)$

$\Rightarrow$ positive x orientation

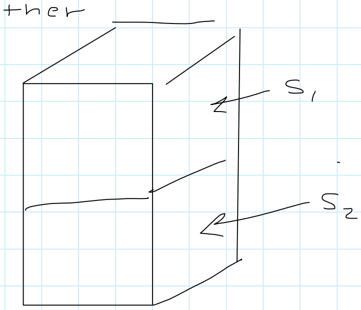
Green's Thm computes line integral of a vector field along a closed curve

Divergence Thm computes the surface of a vector field along a closed surface

eg: sphere, cube, tetrahedron

$\hookrightarrow$ the boundary of a bounded solid in $\mathbb{R}^3$

Idea suppose we stack 2 cubes on top of each other



S_3 is boundary of rectangular prism formed by these cubes.

give all closed surfaces the outward orientation

$\Rightarrow$ on common face btwn 2 cubes, you have opposite orientation

$\Rightarrow$ cancellation of surface integrals

$$\Rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\, dA + \iint_{S_2} \vec{F}\cdot\vec{n}\, dA = \iint_{S_3} \vec{F}\cdot\vec{n}\, dA$$

In general, if we have a closed surface S s.t.

$S = \partial V$

"boundary of"

and we break up V into

$V = V_1 \cup V_2 \cup \dots \cup V_N$

Let $S_i = \partial V_i$, then:

$$\iint_S \vec{F}\cdot\vec{n}\, dA = \sum_{i=1}^{N} \iint_{S_i} \vec{F}\cdot\vec{n}\, dA$$

by the same cancellation idea as with stacked cubes

Idea of Div Thm

compute $\iint_S \vec{F}\cdot\vec{n}\, dA = \iint_S \vec{F}\cdot d\vec{\sigma}$

by breaking V into little pieces & adding them up then approximate the little pieces using derivative approximations for $\vec{F}$.

As the pieces get smaller, the approximation gets better & the sum becomes an integral

Consider a little piece V_i. Say it's a cube with verticies (x_i, y_i, z_i) and $(x_i + \Delta x, y_i, z_i)$ and $(x_i, y_i + \Delta y, z_i)$ and $(x_i, y_i, z_i + \Delta z)$ s.t. $\Delta x = \Delta y = \Delta z$ ($\Rightarrow$ cube)

This cube has 6 faces, which are divided into 3 pairs of opposite corrresponding to 3 coord direction.

e.g. consider the opposite faces in the x-dir

Face 1 (x_i, y_i, z_i), $(x_i, y_i + \Delta y, z_i)$, $(x_i, y_i, z_i + \Delta z)$

$\hookrightarrow$ $-x$ orientation

Face 2 same but shifted in x-direction by Δx

$\hookrightarrow$ $+x$ orientation

b/c normal vector is on x-axis, we care only about the x-coord (aka $\vec{\imath}$-coord) of $\vec{F}$ for this pair of faces bc normal vector is on x-axis, we care only about the x-coord (aka $\hat{i}$-coord) of $\vec{F}$ for this pair of faces

say $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$\Rightarrow \iint_{\text{face 1}}$

$$\Longrightarrow \iint_{\text{face 1}} \vec{F}\cdot d\vec{\sigma} \approx P(x_i, y_i, z_i)\ldots$$

$$\Longrightarrow \iint_{\text{face 2}} \vec{F}\cdot d\vec{\sigma} \approx P(x_i+\Delta x, y_i, z_i)\Delta y \Delta z$$

$$\Longrightarrow \iint_{\text{face 1}} \vec{F}\cdot d\vec{\sigma} + \iint_{\text{face 2}} \vec{F}\cdot d\vec{\sigma}$$

$$= P(x_i+\Delta x, y_i, z_i) - P$$

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$$\Longrightarrow \iint_{\text{face } 1} \vec{F}\cdot d\vec{\sigma} + \iint_{\text{face } 2} \vec{F}\cdot d\vec{\sigma}$$

$$= \left[P(x_i + \Delta x, y_i, z_i) - P(x_i, y_i, z_i)\right]\Delta y \Delta z$$

$$\approx \left[\frac{\partial P}{\partial x}(x_i, y_i, z_i)\Delta x\right]\Delta y \Delta z.$$